A Basic Guide to Super-Symmetry and the Wess-Zumino Model

Richard J Miller

Issue 1.1 14/03/2022

This document gives a basic introduction to super-symmetry, starting with a study of some of its key properties, as can also be elucidated via the simpler theory known as super-symmetric quantum-mechanics, and proceeding to the more advanced field-theory using a simplified form of the Wess-Zumino model.

To keep the article relatively short, given the complexity of the subject, it assumes a post-graduate level of familiarity with relativity and field-theory, with references provided to give a readable background to the material.

Contents

Acronyms and Abbreviations

GR : General Relativity
QFT : Quantum field Theory
SUSY : Super-Symmetry (the full relativistic QFT)
SSQM : Super-Symmetric Quantum Mechanics (a simpler, non-relativistic SUSY)
ST : Spacetime
WZ : Wess-Zumino (SUSY foundation 1974 papers)

Conventions

Natural units ($c = 1, \hbar = 1$) are used throughout

The Minkowski metric is of *mostly minus* form, i.e. $\mathrm{diag}(+, -, -, -)$

Roman indices i, j, k 1 to 3 denote three-vector, spatial coordinates

Greek indices μ, ν, ρ, σ etc. 0 to 3 denote four-vector coordinates

All spinors χ, η, ϵ are left-chiral - right chiral spinors not required

Literature

Three key sources of information were used for this article, namely Junker [1], Labelle [2] and Ryder [3]. The first, Junker [1], is strictly super-symmetric quantum mechanics (SSQM), but provided the impetus to pursue the real-thing, as it were, i.e. super-symmetric field theory (SUSY). For SUSY, the second reference, Labelle [2], is a comprehensive account with no compromising on mathematical content, and goes to a level far in advance of what is required for this article. To supplement SUSY, and for general QFT, the ever-indispensable third reference, Ryder [3], is also useful, especially for the non-SUSY relativistic/Lorentz group details. As always, Penrose [4] provides a good layman-level background (albeit advanced) for such topics as spinors and Grassman algebras. Last, and most definitely not least, the founding Lagrangian formulation of SUSY, as made concrete by the Wess-Zumino Lagrangian, is given in their 1974 papers [5], [6]. These references were only mentioned last since they actually provide a more advanced Lagrangian than the simplified form discussed herein, and are thus not explicitly required.

1 Super-symmetry

1.1 Overview

Super-symmetry aims to unify the bosonic and fermionic aspects of nature in a single mathematical field theory whereby these two apparently distinct aspects are just two sides of a single theory, related to each other by a symmetry transformation. In the physical world there is a natural division of particles into those of integer spin (this includes zero spin), known as bosons, and those of half-integer spin, known as fermions; it is the aim of super-symmetry to produce a common description for both types in a single mathematical framework. However, super-symmetry is not just about particles, because spacetime itself is also considered bosonic, whilst spin-half particles are associated with fermionic fields. Therefore super-symmetry also attempts to unify the bosonic space-time coordinates with these fields in a single *super-space*.

Whilst the idea of super-symmetry might seem a lofty concept, lacking in experimental evidence, the idea of unification of particles and forces on the basis of symmetry underpins the standard model and it is hard to see why nature wouldn't want to keep the same principles in all its manifestations. Notably, with gravity currently resisting all attempts to unify it with the standard model, super-symmetry offers a way forward; it is integral to both *super-string theory*, aimed at unifying all four forces in one go, and the more modest *super-gravity*, that aims to reformulate general relativity as a quantum field theory, from where further advances may then lead to a full unification.

As mentioned above, there is currently no experimental evidence to support super-symmetry. Indeed, as will be discussed herein, one of the main predictions of super-symmetry is that every boson has a super-symmetric fermion partner, and vice versa, with the same mass [1]. However, such partners have not been seen so far and so it is thought that, if they exist, then they must have a mass much greater than their standard-model partners, with the extra mass acquired by a symmetry-breaking mechanism. In favour of this argument, such a mechanism is not without theoretical precedent, and backed-up by experimental verification, namely the discovery at CERN, starting in 1982, of the three gauge bosons $(W\pm, Z^0)$ of the weak force, that were originally predicted as massless until a symmetry-breaking mechanism was incorporated into the electroweak theory [2]. On a more general, but related note, the last 100 years have seen numerous physical predictions based solely upon the mathematical theory, and only experimentally verified many years or decades later. Witness GR's prediction of gravitational waves and black-holes, and Dirac's prediction of anti-particles.

In brief, if the mathematical theory predicts it, nature seems to follow it, and super-symmetry is therefore likely too good not to pursue.

1.2 Introduction

Whilst this article primarily concerns SUSY as a quantum field theory (QFT), via a simplified variant of the *Wess-Zumino* Lagrangian, the basic principles of SUSY are also encapsulated in a simpler theory known as super-symmetric quantum mechanics [1] (SSQM), which is non-relativistic, unlike QFT. With this in mind, the general results given next can also be seen in more readable SSQM texts.

At the heart of super-symmetry is a single charge operator Q and its conjugate $Q^\dagger$ that both act on fermion states ψ_F and boson states ψ_B as follows:

$$\begin{aligned} \text{fermion to boson} \quad & Q\psi_F = \psi_B \\ \text{boson to fermion} \quad & Q^\dagger\psi_B = \psi_F \\ \text{boson annihilation} \quad & Q\psi_B = 0 \\ \text{fermion annihilation} \quad & Q^\dagger\psi_F = 0 \end{aligned} \tag{1}$$

From these equations it is easily seen that

$$\begin{aligned} Q^\dagger(Q^\dagger\psi_B) &= Q^\dagger\psi_F = 0 \\ Q(Q\psi_F) &= Q\psi_B = 0 \end{aligned} \tag{2}$$

i.e. Q and $Q^\dagger$ both square to zero

$$Q^2 = {Q^\dagger}^2 = 0 \tag{3}$$

[1] that super-symmetry makes testable predictions with current technology is laudable in itself. Unfortunately, so far, current technology is just constraining the super-partner masses to ever higher energies

[2] The theoretical unification of the weak force and electromagnetism, known as 'electroweak', was completed in the 1970s, and, together with QCD, essentially wrapped-up the Standard Model. Glashow, Salam and Weinberg received the 1979 Nobel prize in physics for the electroweak theory, and experimental evidence in the form of the $W\pm$ and Z^0 gauge bosons with their predicted masses (acquired due to symmetry breaking), was provided in stages, starting three years later in late 1982 at CERN. This was, indeed, a triumph for the laboratory, all those behind QED going right back to the 1930s, and a great source of inspiration for the author of this article!

In essence then, Q and $Q^\dagger$ are two-state operators that flip a fermion to a boson and vice-versa. Of course, this action is identical to that of two-state spin, raising and lowering operators. Note that, by comparison, whilst the harmonic oscillator raising operator raises the energy state by an equal amount ($\hbar/2$) upon every consecutive application, unbounded in the number of applications, the super-symmetry charges do not only takes two consecutive applications to annihilate the original state (fermion or boson). These are known as quadratic nilpotent operators (matrices) in linear algebra.

Quantities that square to zero satisfy the following anti-commutation rules, and such anti-commuting numbers are also known as Grassman numbers, Appendix (4.2):

$$\begin{aligned} \{Q, Q\} &= 0 \\ \{Q^\dagger, Q^\dagger\} &= 0 \end{aligned} \tag{4}$$

There remains the mixed anti-commutator $\{Q, Q^\dagger\}$, which actually relates to the zeroth component of the four-momentum, i.e. the energy or Hamiltonian in quantum mechanics, as in

$$\{Q, Q^\dagger\} = P^0 = E = H \tag{5}$$

Note that SUSY works in terms of four-momentum P^μ, but the simpler SSQM works simply in terms of the energy $E = P^0$.

Anti-commutation is key to the SUSY charges, and additional to standard model QFT, which works with charges (generators of a Lie algebra [3], [7]) that are defined through their commutation relations.

From the anti-commutation relation (5) it can be shown that the ground state energy is zero. To see this, consider an energy eigenvector state ψ of the Hamiltonian H, then the energy expectation $\langle E\rangle$ is given by $\psi^\dagger H\psi$, and using (5) to expand H in terms of $Q, Q^\dagger$, gives

$$\begin{aligned} \langle E\rangle &= \psi^\dagger H\psi = \\ &= \psi^\dagger(QQ^\dagger + Q^\dagger Q)\psi \\ &= \psi^\dagger QQ^\dagger\psi + \psi^\dagger Q^\dagger Q\psi \\ &= (Q^\dagger\psi)^\dagger(Q^\dagger\psi) + (Q\psi)^\dagger(Q\psi) \\ &= (Q^\dagger\psi)^2 + (Q\psi)^2 \geq 0 \end{aligned} \tag{6}$$

Since both terms on the last line are zero or positive, then the lowest energy state, i.e. that of the ground state, is never less than zero. Furthermore, since each square term is zero or positive, then it is seen that both Q and $Q^\dagger$ annihilate the ground state ψ_0, i.e.

$$H\psi_0 = 0 \Leftrightarrow Q^\dagger\psi_0 = Q\psi_0 = 0 \tag{7}$$

Whilst anti-commutation is an important property of the charges, so too is the standard commutation of the charges with all components of the four-momentum, i.e.

$$[Q, P^\mu] = [Q^\dagger, P^\mu] = 0 \tag{8}$$

This commutation property leads to a very important fact, namely that the mass (Lorentz invariant rest mass m) is the same for both the fermion and and its boson super-partner [3].

[3] This is a relativistic SUSY result, but the non-relativistic SSQM equivalent is equivalence in energy, i.e. for a non-zero energy, the fermion and boson state is energetically degenerate. In both SUSY and SSQM, the spin is not degenerate, of course, since fermions are half-integral and bosons integral.

To see this, firstly note that the squared momentum operator P^2 gives the squared mass of a state, e.g., for a fermion,

$$P^2\psi_F = m^2\psi_F \tag{9}$$

Operating on both sides with Q, and using the fact that Q commutes with the scalar mass m, then

$$QP^2\psi_F = m^2 Q\psi_F \tag{10}$$

and using (1) on the right this becomes

$$QP^2\psi_F = m^2\psi_B \tag{11}$$

Given that P^2 commutes with Q, since P^μ commutes with Q, i.e.

$$[P^2, Q] = 0 \tag{12}$$

then

$$QP^2\psi_F = P^2 Q\psi_F = m^2\psi_B \tag{13}$$

and finally applying (1) to the middle term shows that

$$P^2\psi_B = m^2\psi_B \tag{14}$$

And so, if ψ_F has mass m (9) then so too does ψ_B, i.e. **the fermion has the same mass as its super-symmetric boson partner**.

This is a major conclusion, but experimental evidence says otherwise since such equivalent mass partners for each of the known particles has not been found. Hopefully, somehow, just like electroweak, the fermion/boson symmetry is broken.

To summarise all the above results, super-symmetry has the following properties:

1) The charges mix bosons and fermions

2) The charges anti-commute

3) The ground state energy is greater than or equal to zero

4) The charges annihilate the ground state

5) The fermion and boson masses are identical

The above frames super-symmetry in terms of its charges - or rather *generators* as they are more commonly known (since they generate the symmetries). However, the explicit form of the generators Q and $Q\dagger$ has never actually been given, only their algebra, as per their the commutation relations amongst themselves and with the generators of other symmetries, notably here the four-vector momentum acting in its role as a generator (of translations).

The full SUSY

As stated, the above description is a simplified version of super-symmetry more akin to SSQM, but the same results hold for a full SUSY QFT. In particular, the algebra of the super-charges is expanded in a full SUSY to encompass the other key generators, notably the spacetime generators of the Poincare group [3].

In standard QFT, the 10-generator Poincare group of spacetime symmetries is formed from the six generators of the Lorentz group $M_{\mu\nu}$ (a linear combination of three boosts K_i and three rotations J_i) plus the four momentum generators P_μ. The SUSY fermionic charges (generators) Q add to this set of Poincare generators (as too the internal symmetry generators for such internal quantum symmetries as isospin, parity etc.). Indeed, the generators do not just 'add', but effectively mix via anti-commutation relations as given next.

Denoting a single supercharge Q by a left-chiral spinor, comprising two-elements $Q = (Q_1, Q_2)$, component index α (or β), i.e. Q_α (or Q_β) where $\alpha, \beta \in 1, 2$, then the charges and Poincare generators satisfy the following anti-commutation relations

$$\begin{aligned} \{Q_\alpha, Q_\beta\} &= \{Q_\alpha^\dagger, Q_\beta^\dagger\} = 0 \\ \{Q_\alpha, Q_\beta^\dagger\} &= 2(\sigma^\mu)_{\alpha\beta} P_\mu \\ \{Q_\alpha, M_{\mu\nu}\} &= \frac{1}{2}(\sigma_{\mu\nu})_\alpha^\beta Q_\beta \end{aligned} \tag{15}$$

The charges also satisfy the following commutation relations:

$$\begin{aligned} [Q_\alpha, P_\mu] &= 0 \\ [Q_\alpha, M_{\mu\nu}] &= (\sigma_{\mu\nu})_\alpha^\beta Q_\beta \end{aligned} \tag{16}$$

These results are quoted without proof, see [2].

The above relations pertain to a single super-charge Q. However, there can be many super-charges Q^i, $i = 1 \dots N$, and then the commutation relations expand, notably

$$\{Q_\alpha^i, Q_\beta^{j\dagger}\} = 2\delta^{ij}(\sigma^\mu)_{\alpha\beta} P_\mu, \quad i, j \in 1 \dots N \tag{17}$$

where the σ^μ are the four Pauli matrices, given below for quick reference

$$\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

together with four-vectors σ and $\bar{\sigma}$ (a vector of Pauli matrices) defined as as

$$\sigma = (\sigma_0, \sigma_i), \quad \bar{\sigma} = (\sigma_0, -\sigma_i)$$

Whilst there are a few common variants of SUSY notably $N = 4$, only the single super-charge case for Q and its transpose conjugate $Q^\dagger$ is considered herein as that is all that is necessary to illustrate the key aspects of SUSY.

1.3 The Lagrangian Method

As currently known, the laws of nature, e.g. the equations governing the dynamical interaction of particles and fields, such as gravity and electromagnetism, are all derivable from an action principle. The *action* S (below) is given as the integral over all four spacetime coordinates x_μ of a Lagrangian density function $\mathcal{L}$, where the Lagrangian itself is generally a function of the spacetime coordinates, the fields ϕ_a (index a goes from 1 to the number of fields) and their derivatives $\partial_\mu \phi_a$ up to at least second order, e.g. $\partial_{\mu\nu}\phi_a$.

$$S = \int_\Omega \mathcal{L}(x_\mu, \phi_a, \partial_\mu \phi_a, \partial_{\mu\nu}\phi_a) dx^4 \tag{18}$$

The field ϕ here is generic and could represent either bosonic fields, as per those associated with the gauge bosons (photon, $W\pm$, Z^0 and gluons), or fermionic fields associated to the fundamental particles, i.e. leptons and quarks, that are all spin 1/2 fermions. Note that the gravitational field is bosonic and, if a quantum theory of gravity exists, then its gauge particle is the spin-2 graviton - a boson.

The action principle states that the field equations are such that any variation in the action δS due to a functional change in the Lagrangian $\delta\mathcal{L}$, via changes to the fields $\delta\phi_a$ and their derivatives, $\delta\partial_\mu\phi_a$, is a stationary point, i.e. $\delta S = 0$ as in [4]

$$\delta S = \int_\Omega \delta\mathcal{L} = 0 \tag{19}$$

and the functional variation $\delta\mathcal{L}$ of the Lagrangian is of the following form:

$$\delta\mathcal{L} = \frac{\partial\mathcal{L}}{\partial\phi_a}\delta\phi_a + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)}\delta(\partial_\mu\phi_a) + \frac{\partial\mathcal{L}}{\partial(\partial_{\mu\nu}\phi_a)}\delta(\partial_{\mu\nu}\phi_a) \tag{20}$$

Not surprisingly, super-symmetry is also developed in terms of a Lagrangian and action principle, as now detailed via a simple, but non-trivial, model.

Generators

The fundamental field variation $\delta\phi$ is specified by the generators of the group of whichever transformations it is desired to apply, i.e. in SUSY, these generators will be the charges Q_i.

As a familiar example, if P_μ is the generator of translation, then an operator U (unitary in this and general quantum mechanics) is constructed from it in the following, standard way:

$$U(a^\mu) = exp(ia^\mu P_\mu/\hbar) \tag{21}$$

and if ϕ is a simple scalar field then it transforms under this unitary operator U by simple multiplication as in

$$\phi \rightarrow U\phi \Rightarrow \delta\phi = U\phi - \phi \tag{22}$$

Invariably, in QFT, the field ϕ is an operator, say $\hat{\phi}$, and then the field transforms as follows, which is a standard operator transform

$$\hat{\phi} \rightarrow U\hat{\phi}U^\dagger \Rightarrow \delta\hat{\phi} = U\hat{\phi}U^\dagger - \hat{\phi} \tag{23}$$

Up to this point no requirement has been placed on the translation a^μ being of small magnitude and, consequently, neither is the field change $\delta\phi$ necessarily small. However, if a^μ is small, then the exponential can be expanded to first order in terms of the generator P_μ itself, and the corresponding small field change is then just a linear function of the generators, i.e. for scalar field ϕ

$$\delta\phi = -ia^\mu\frac{P_\mu}{\hbar}\phi, \quad |a^\mu| \ll 1 \tag{24}$$

The important point being that, in general, the infinitesimal field transformations are dictated by the generators and, as oft stated, finite transformations can always be constructed from an infinite sequence of infinitesimal changes, as algebraically constructed by the exponentiation of the generator.

For QFT and SUSY, the fields are invariably operators such as $\hat{\phi}$, and then, for small translations (keeping with the above example), the above small change $\delta\hat{\phi}$ is given by a commutator, to first order, as follows:

$$\delta\hat{\phi} = -ia^\mu[P_\mu/\hbar, \hat{\phi}] \tag{25}$$

[4] Note that $\delta S = 0$ does not mean $\delta\mathcal{L}$ is necessarily zero (although it often is). Indeed, it is sufficient that $\delta\mathcal{L}$ is only unique to within a total derivative of a function F as in, $\mathcal{L} \rightarrow \mathcal{L} + \partial_\mu F$

This is also still a nice linear function of the generator but now, very importantly (very), the field transformation is a commutator and so it can be understood why mathematical physics seems to give such importance to the generators and their commutator algebra, which is ultimately that due to Lie [5] - the generators satisfy a *Lie algebra* as defined by their commutation relations [7]. Note that the operators formed from the exponentiation of the generators form the Lie group, but since the group elements are intimately linked to the generators, the term group often gets used when referring to the generators, as in 'the Lie group of generators' - harmless but confusing to a novice!

The reader may well be wondering what this all has to do with SUSY, but the charges Q and $Q^\dagger$ are also generators and, indeed, they generate the symmetry group of transformations that flip bosons into fermions and vice versa, as stated earlier in the Introduction.

Aside. Because the SUSY generators square to zero (see the Introduction), the expansion of the exponentiated operator form e^{iaQ}, for arbitrary constant a, only has a first order expansion since evidently all terms $(iaQ)^2$ and higher order are zero, i.e. the expansion $e^{iaQ} = 1 + iaQ$ is exact. Although this is not a general property (to the author's knowledge) shared by standard-model generators, and purely a consequence of the SUSY charges anti-commutation, Grassmann algebra, it is noted by the author that the spin raising and lowering operators (e.g. $\sigma_\pm = \sigma_1 \pm i\sigma_2$) have this quadratic nilpotency, i.e. they square to zero ($\sigma_\pm^2 = 0$). Because of this, they can be used as a 2x2 matrix representation of the charges Q and $Q^\dagger$ in SSQM. Neither has it escaped the authors attention that quadratic nilpotency is a feature of the differential of an exact form being zero, i.e. $d^2\omega = 0$. This is more formally known as the *the Poincare Lemma* that *all exact forms are closed*, and also has the geometric interpretation that the boundary of a boundary is zero - see the *coboundary operator* in [3], also Penrose [4] on the subject of differential geometry. It is believed this connection (no geometric pun intended) between the charges may already have been put to use in SUSY, or at least noted but, once again, is beyond the authors knowledge.

The subject of the SUSY charges as generators is revisited again later in Section (1.7) when discussing variations in the spinor field χ, which is a new field incorporated into a classical free-field Lagrangian to give a working SUSY Lagrangian, as detailed next.

1.4 A Simplified Wess-Zumino Model

The study of super-symmetry starts by defining a simple, but non-trivial Lagrangian containing bosonic (scalar/tensor) and fermionic (spinor) fields whose action is invariant under transformations which convert one field type to the other Such a Lagrangian is defined as follows:

$$\mathcal{L} = \partial_\mu \phi \partial^\mu \phi^\dagger + \chi^\dagger i \bar{\sigma}^\mu \partial_\mu \chi \tag{26}$$

This Lagrangian (strictly speaking Lagrangian *density*) is a much-simplified version of that originally presented in the 1974 Wess-Zumino paper [5], and comprises a single, complex, scalar field ϕ and a two-spinor χ, both functions of four spacetime coordinates x^μ. The complex scalar field ϕ is bosonic (integer spin), e.g. Klein-Gordon, and the spinor field χ is fermionic (half-integer spin), e.g. Dirac.

Whilst the above Lagrangian is a much simplified form of the full Wess-Zumino (WZ) model, it serves to illustrate all the important aspects of a super-symmetric theory. Indeed, this is a fully relativistic (manifestly covariant [6]) Lagrangian density, and the corresponding action is

[5] Sophus Lie 1842-1899, see Penrose [4]

[6] strictly speaking this is only true for special relativity, not general relativity, which requires covariant, not partial, derivatives). However, this article most definitely only pertains to special relativity

thus integrated over all four ST coordinates x_μ

$$S = \int \mathcal{L} dx^4 \tag{27}$$

The Lagrangian has no mass term [7] and hence both fields ϕ and χ are massless. Neither are there any interaction terms between the fields or other, and hence it is known as a *free-field* model.

The complex scalar field can be split into two real-valued fields, as is often seen in texts [5], [3], e.g. $\phi = A + iB$ and $\phi^\dagger = A - iB$ for some real scalar fields A and B. However, this split is not used herein.

There is only one fermionic field χ, and hence this simple model is thus known as an $N = 1$ super-symmetric model. Further afield, the more-general $N = 1$ case is known as the *Minimally Super-Symmetric Model* (MSSM), which is more advanced than the simple model given above.

Both fields ϕ and χ are scalar functions here, not operators, and, accordingly, the treatment of the simplified WZ model hereafter is essentially classical field theory, i.e. not canonical (or second) quantisation, as would befit a full quantum-field theoretic treatment of super-symmetry. This simplification keeps the algebra relatively straightforward and at a readable level for graduate physicists and mathematicians.

Keep in mind that, all the time, bosonic quantities commute, whereas fermionic quantities anti-commute, i.e.

$$[\phi, \phi] = 0, \quad \{\chi, \chi\} = 0 \Rightarrow \chi^2 = 0 \tag{28}$$

As mentioned earlier, this anti-commuting algebra is a key property of the SUSY charges Q and $Q^\dagger$, and also the infinitesimal fermion spinor variations ϵ and $\epsilon^\dagger$, as introduced shortly when varying the action. Note that symbol ϵ here denotes a two-spinor, and not the pseudo-metric also used in super-symmetry to raise/lower spinor indices (and neither is ϵ the commonly-used Levi-Cevita symbol either).

The fermionic term is effectively one-half of the Dirac equation [3] for the left-chiral *Weyl* two-spinor χ, when the Dirac equation is presented in the *chiral representation* for the massless case. In this case, the Dirac-equation decouples into the direct sum, SU(2)$\oplus$SU(2), of what is termed a left and right part. Note that left chiral, as opposed to right, is only chosen by convention, with this convention largely due to neutrinos coming only in left-handed variants - at least when massless.

The fermionic term is asymmetric in χ in that the derivative only acts on χ, not $\chi^\dagger$. This could be remedied by symmetrisation of the term, i.e. splitting into the following form

$$\frac{1}{2}(\chi^\dagger i\bar{\sigma}^\mu \partial_\mu \chi + \partial_\mu \chi^\dagger i\bar{\sigma}^\mu \chi) \tag{29}$$

but it is harmless to leave the Lagrangian asymmetric, it just means the field equations will not have complex-conjugate symmetry.

Since the integration measure dx^4 is of mass dimension -4 (Appendix 4.1) then the Lagrangian density is accordingly of dimension 4 so that the action is a scalar by definition. Ensuring the mass dimension is consistent makes for a useful check on the algebra that will follow when next studying the field variations, $\delta\phi$ etc.. Indeed, a consistent mass dimension acts as a very useful

[7] If the scalar field ϕ were massive, there would be an extra term $m^2\phi^\dagger\phi$, quadratic in mass and ϕ, hence with a total mass dimension 4 in accordance with the mass dimension of the Lagrangian.

guide when constructing variational terms, albeit such construction will not be manually done here and they will be quoted as per [2]. Looking at the complex scalar field term in $\mathcal{L}$, with reference to the dimensional values in Appendix (4.1), it is confirmed to have a dimension 4 since $[\partial_\mu \phi] = [\partial^\mu \phi] = 2$. For the fermionic part this is slightly less trivial because the $[\bar{\sigma}]$ term is a dimensionless generator, and the χ terms are dissimilar. Nevertheless, since $[\partial_\mu] = 1$, and $[\chi] = [\chi^\dagger]$, this means $[\chi]$ is fractional equal to 3/2 with $[\partial_\mu \chi] = 5/2$

1.5 Field Equations

The boson and fermion fields in the Lagrangian (26) are non-interacting and, taken in isolation, the Lagrangian splits trivially into two separate forms, one for each field. Taking variations with respect to each then gives their individual field equations as

$$\frac{\delta S}{\delta \phi} = \partial^\mu \partial_\mu \phi^\dagger = 0, \quad \frac{\delta S}{\delta \phi^\dagger} = \partial^\mu \partial_\mu \phi = 0 \tag{30}$$

$$\frac{\delta S}{\delta \chi^\dagger} = i\bar{\sigma}^\mu \partial_\mu \chi = 0, \quad \frac{\delta S}{\delta \chi} = i\partial_\mu \chi^\dagger \bar{\sigma}^\mu \tag{31}$$

For the field ϕ, the field equation is just the standard second order wave equation (Klein-Gordon equation for a massless particle), where p^μ is the usual four-momentum

$$\phi = \exp(ip^\mu x_\mu) \tag{32}$$

and the for the spinor field χ, the field equation is just the left-chiral half of the first-order Dirac equation for massless particles

$$\sigma_i \cdot p^i \chi = -p^0 \chi, \quad E = p^0 = |\bar{p}|, \bar{p} = p^i \tag{33}$$

where $\sigma_i \cdot p^i/|\bar{p}|$ is the helicity operator, and so it can be seen from this that χ is an eigenspinor of this operator for eigenvalue -1, i.e. a left-chiral spinor (right chiral has eigenvalue +1).

Aside. a right chiral spinor has a spin vector pointing in the direction of travel, whereas it is in the opposite direction of travel for a left-chiral spinor. The 'left/right' nomenclature is, of course, just convention, but in widespread use. Well-defined helicities are strictly only applicable to massless particles i.e. the photon (a boson) and neutrino (a fermion) - the latter now suspected to be massive (albeit < 0.5eV by current experimental data). Neutrinos are theorised to only come in left-chiral forms - an asymmetry due too the weak-force, and the main reason why super-symmetry is formulated only using left-chiral spinors.

1.6 Auxiliary Fields

The above solution in χ, i.e. $i\bar{\sigma}^\mu \partial_\mu \chi = 0$ and its conjugate, is termed *on-shell* (AKA *physical* solutions) because its solution satisfies the masssless, energy momentum equation, i.e. $(p^0)^2 - |\bar{p}|^2 = 0$. However, in the world of Quantum field theory, not least SUSY, off-shell solutions (*non-physical*) are possible and accounted for in the computations. The important point here is that restricting only to on-shell solutions in the analysis that follows limits the applicability of the equations and is basically another simplification that a full SUSY would not have. The limitation is actually removed by adding a new auxiliary field F to the Lagrangian, a term of the form $FF^\dagger$, so that the Lagrangian 26 now becomes,

$$\mathcal{L} = \partial_\mu \phi \partial^\mu \phi^\dagger + \chi^\dagger i\bar{\sigma}^\mu \partial_\mu \chi + FF^\dagger \tag{34}$$

The auxiliary field F will not be pursued much further since it also adds more complication to the algebra that follows in the next section. However, invariably readers will encounter the use of auxiliary fields in the literature [2], [3] and [5], and a few comments therefore follow to finish this section.

The field is scalar, bosonic, with mass dimension [2] to make for a total mass dimension of the term $[FF^\dagger] = 4$, consistent with all terms in the Lagrangian,

Since it is really two fields, F and $F^\dagger$ as per ϕ and χ, it might therefore be seen in literature in the more symmetric form, i.e. $(FF^\dagger + F^\dagger F)/2$ or, indeed, split into two fields, e.g. Ryder [3] uses $F + G$ without conjugate (and also A and B for ϕ), and the founding paper [5] also takes this approach.

Given the field's introduction is primarily to overcome a limitation, its simplistic form is to be welcomed, i.e. the simpler the better when it comes to adding new terms. However, to restore the invariance of the action, as studied next, its presence also requires additional terms in the field variations, (35) below, and so it will now be dropped as unnecessary obfuscation.

To close this matter with some more positive news, it seems that the auxiliary fields can at least be replaced by a much more physically meaningful term known as the *super-potential*, which is a potential-like function $\mathcal{W}$ of the scalar field ϕ, see [2] for more information.

1.7 Variational Methods

Consider the following variational terms, [2], where the small variational parameter is a left-chiral two-spinor ϵ

$$\delta\phi = \epsilon \cdot \chi = \epsilon^T(-i\sigma^2)\chi \tag{35}$$

$$\delta\phi^\dagger = \bar{\epsilon} \cdot \bar{\chi} = \chi^\dagger i\sigma^2 \epsilon^* \tag{36}$$

$$\delta\chi^\dagger = -i\partial_\mu \phi^\dagger \epsilon^T i\sigma^2 \sigma^\mu \tag{37}$$

$$\delta\chi = -i\partial_\mu \phi \, \sigma^\mu i\sigma^2 \epsilon^* \tag{38}$$

These variational terms look rather foreboding, particularly the last two. However, the first two are spinor inner products as defined in Appendix (4.2), and the last two will be given some more explanation shortly.

The variational terms go straight to the heart of super-symmetry since all four mix the fermionic field χ with the scalar complex field ϕ. As can be seen above, the first two relate a variation in bosonic field ϕ to the spinor field χ, and the last two give the converse, albeit in terms of the ST derivatives of ϕ.

The above transformation of all fields is taken to be a global, not local, and the variational spinor ϵ is therefore not a function of spacetime, i.e.

$$\partial_\mu \epsilon = 0 \tag{39}$$

This simplification means that the above model does not give a proper gauge-field theory, albeit SUSY most definitely does have local gauge invariance. Again, this restriction to a global variation is a simplification that does not diminish the key aspects of SUSY.

Because ϵ is a complex spinor (with Grassman algebra), there are really two independent spinor variations ϵ and $\epsilon^\dagger$, i.e. $\epsilon = \epsilon_1 + i\epsilon_2$ and $\epsilon^\dagger = \epsilon_1 - i\epsilon_2$, for two, spinors ϵ_1, ϵ_2. As a consequence, there are two independent variational terms in the variation of the action, one in ϵ and one in $\epsilon^\dagger$

All the variational terms are constructed to adhere to Lorentz invariance, linearity and, in addition, have a consistent mass-dimension, whilst attempting to make them as simple as possible. Linearity in the fields and their derivatives is a key guiding principle since non-linear terms make for a complicated non-linear theory, which should only be invoked when all attempts at a linear theory fail - fortunately this is not the case. As regards Lorentz invariance, the first two terms are evidently Lorentz invariant since they are constructed from Lorentz invariant inner products Appendix (4.2). The last two terms are more cryptic, but the scalar field derivatives ($\partial_\mu \phi$ and $\partial_\mu \phi^\dagger$) are manifestly covariant (and linear), and the remaining quantities ($\epsilon^T i\sigma^2\sigma^\mu$ and $\sigma^\mu i\sigma^2\epsilon^*$) are also of an inner product form, given ϵ and σ (in various guises) are sandwiched between a spinor form of *metric* - Appendix (4.2).

As regards consideration of mass dimension, taking the simple first variation ($\delta\phi = \epsilon \cdot \chi$) as an example, then, since the mass dimension $[\phi] = 1$ and $[\chi] = -3/2$, Appendix (4.1), the mass dimension of ϵ is evaluates to $-1/2$. The remaining variational terms are also seen to be consistent using the dimension values given in Appendix (4.1).

Applying the variations

Given the above variations, the next step is to show these do indeed leave the action invariant ($\delta S = 0$), i.e. they are a symmetry transformation on the Lagrangian that gives the same field equations. In fact, it will be seen that the variations actually also leave the Lagrangian invariant as in $\delta\mathcal{L} = 0$, which is a stricter constraint - a general variation need only leave the Lagrangian invariant to within a total derivative.

Taking small 'δ' variations of the fields in the Lagrangian gives

$$\delta\mathcal{L} = (\partial_\mu \delta\phi)\, \partial^\mu \phi^\dagger + \partial_\mu \phi\, (\partial^\mu \delta\phi^\dagger) + (\delta\chi^\dagger) i\bar{\sigma}^\mu \partial_\mu \chi + \chi^\dagger i\bar{\sigma}^\mu (\partial_\mu \delta\chi) \tag{40}$$

and using the above variational terms (35), with some μ, ν index swapping, the variational terms become

$$\partial_\mu \delta\phi = \epsilon^T(-i\sigma^2)\partial_\mu \chi \tag{41}$$
$$\partial^\mu \delta\phi^\dagger = \partial^\mu \chi^\dagger i\sigma^2 \epsilon^* \tag{42}$$
$$\delta\chi^\dagger = -i(\partial_\nu \phi^\dagger)\epsilon^T i\sigma^2\sigma^\nu \tag{43}$$
$$\partial_\mu \delta\chi = -i(\partial_\mu \partial_\nu \phi)\, \sigma^\nu i\sigma^2 \epsilon^* \tag{44}$$

Note that because ϵ is global then, as above, there are no derivative terms $\partial_\mu \epsilon$, which means there is no need to integrate-by-parts $\partial_\mu \epsilon$ terms to get back to terms linear in ϵ - for local ST transformations this process is the norm since $\partial_\mu \epsilon \neq 0$ (hence the reason for making this point). This is also a good reason to always start with a global variation as a first check, because if that fails, algebraically, then a local variation will be guaranteed to fail.

Substituting the variational terms into the Lagrangian gives

$$\delta\mathcal{L} = \epsilon^T(-i\sigma^2)\partial_\mu \chi\, \partial^\mu \phi^\dagger + \partial_\mu \phi\, (\partial^\mu \chi^\dagger i\sigma^2 \epsilon^*) + \tag{45}$$
$$- i(\partial_\nu \phi^\dagger)\epsilon^T i\sigma^2 \sigma^\nu) i\bar{\sigma}^\mu \partial_\mu \chi + \chi^\dagger i\bar{\sigma}^\mu(-i(\partial_\mu \partial_\nu \phi)\, \sigma^\nu i\sigma^2 \epsilon^*) \tag{46}$$

Because the small spinor variation comes in two independent parts ϵ and $\epsilon^\dagger$, then this can be split into the two independent parts. Doing this gives the two terms:

First and third term in ϵ^T

$$\delta\mathcal{L}_{\epsilon^T} = \epsilon^T(-i\sigma^2)\partial_\mu \chi\, \partial^\mu \phi^\dagger + (-i(\partial_\nu \phi^\dagger)\epsilon^T i\sigma^2 \sigma^\nu) i\bar{\sigma}^\mu \partial_\mu \chi \tag{47}$$

Second and fourth term in ϵ^*

$$\delta\mathcal{L}_{\epsilon^*} = \partial_\mu \phi\, (\partial^\mu \chi^\dagger i\sigma^2 \epsilon^*) + \chi^\dagger i\bar{\sigma}^\mu(-i(\partial_\mu \partial_\nu \phi)\, \sigma^\nu i\sigma^2 \epsilon^*) \tag{48}$$

The scalar field ϕ terms commute with the spinor field χ terms, including derivatives, so too real and complex numbers commute with the spinors, so the two sets of terms are rearranged to give all bosonic, commuting terms on the left, and spinor terms on the right, including Pauli matrices

Looking at the first and third term in ϵ^T, this is now integrated by parts on the spinor terms $\partial_\mu\chi$ to transfer the derivative to the bosonic field ϕ, that now becomes a second order derivatives as follows, where the second term has been reshuffled for what comes next:

$$\delta\mathcal{L}_{\epsilon^T} = i\partial_\mu\partial^\mu\phi^\dagger\epsilon^T\sigma^2\chi + i\epsilon^T i\sigma^2 i\sigma^\nu\bar{\sigma}^\mu\partial_\mu\partial_\nu\phi^\dagger\chi \tag{49}$$

The term $\sigma^\nu\bar{\sigma}^\mu\partial_\mu\partial_\nu\phi^\dagger$ can now also be transformed using the following relation, see [2]:

$$\sigma^\nu\bar{\sigma}^\mu\partial_\mu\partial_\nu\phi^\dagger = \partial_\mu\partial^\mu\phi^\dagger \tag{50}$$

to give

$$\delta\mathcal{L}_{\epsilon^T} = i\partial_\mu\partial^\mu\phi^\dagger\epsilon^T\sigma^2\chi + i\epsilon^T i\sigma^2 i\partial_\mu\partial^\mu\phi^\dagger\chi \tag{51}$$

Lastly, clearing two of the three imaginary units on the right, and reshuffling the additional second order derivative shows that the terms cancel to give a zero variation for ϵ^T, i.e.

$$\delta\mathcal{L}_{\epsilon^T} = i\partial_\mu\partial^\mu\phi^\dagger\epsilon^T\sigma^2\chi - i\partial_\mu\partial^\mu\phi^\dagger\epsilon^T\sigma^2\chi = 0 \tag{52}$$

Now for the second and fourth variational terms in ϵ^*, reproduced below with a little tidying

$$\delta\mathcal{L}_{\epsilon^*} = \partial_\mu\phi\,\partial^\mu\chi^\dagger i\sigma^2\epsilon^* + \chi^\dagger\bar{\sigma}^\mu\sigma^\nu\partial_\mu\partial_\nu\phi\, i\sigma^2\epsilon^* \tag{53}$$

Not surprisingly, the same tricks are performed as above for the first and third terms. The first term on the right is integrated by parts on $\partial_\mu\phi$ to give

$$\delta\mathcal{L}_{\epsilon^*} = -\partial_\mu\partial^\mu\phi\,\chi^\dagger i\sigma^2\epsilon^* + \chi^\dagger\bar{\sigma}^\mu\sigma^\nu\partial_\mu\partial_\nu\phi\, i\sigma^2\epsilon^* \tag{54}$$

and the relation

$$\bar{\sigma}^\mu\sigma^\nu\partial_\mu\partial_\nu\phi = \partial_\mu\partial^\mu\phi \tag{55}$$

substituted into the second term gives, with a slight reshuffle,

$$\delta\mathcal{L}_{\epsilon^*} = -\partial_\mu\partial^\mu\phi\,\chi^\dagger i\sigma^2\epsilon^* + \partial_\mu\partial^\mu\phi\,\chi^\dagger i\sigma^2\epsilon^* \tag{56}$$

Once again, it is seen that both terms cancel so that the ϵ^* variation is also zero, and therefore the total Lagrangian variation is zero, i.e.

$$\delta\mathcal{L} = \delta\mathcal{L}_{\epsilon^T} + \delta\mathcal{L}_{\epsilon^*} = 0 \tag{57}$$

Thus, finally, it has been shown that the Lagrangian (26) is invariant to the variations (35). These variational terms thus represent a symmetry transformation on the Lagrangian and the next step would ordinarily be to identify the generators that generate this symmetry. However, to keep this article to a relatively short length, this is deferred to the literature [2], and only a few closing comments made instead.

Generators and Currents

As discussed earlier in Section (1.3), the field variations are actually generated by 'generators' or charges, but the above started with the field variations stated as is, with no mention of the charges Q that gave rise to the terms. The reason this was done is because [2] gives a very good heuristic derivation of the variational terms. It is actually not easy to obtain an explicit form of the charges and, sadly, beyond the scope of this article. Indeed, as is stated in many field-theory texts, it is not so much the explicit representation of the charges that is of interest, but rather the commutation algebra of these charges, particularly with regard to their mixing with the spacetime generators of the Poincare group. These mixed commutation/anti-commutation relations can be obtained without knowing the explicit generator forms, see [2] for example, and are consequently just stated, without proof, in the Introduction.

That said, a standard method to obtain the charges, as differential operators, is to first calculate the conserved Noether currents of the Lagrangian, and then integrate the zeroth (temporal) component of the four-vector current over the spatial volume d^3x. This is actually not too difficult but, once again, to keep this article relatively short, the reader is referred to [2] where, in addition, the general algebra of the SUSY charges is also given extensive treatment.

Having thus deferred the next steps to the literature, it remains to just summarise the simplifications made, and highlight their remedy and further advancement.

1.8 What next?

The super-symmetric Lagrangian (26) is considered the simplest, non-trivial example required to illustrate all the important concepts of SUSY, whilst keeping the algebra to a minimum. The key simplifications made, and their self-evident remedy, are discussed below.

The symmetry transformation of the fields is global, not local, i.e. the small spinor variation ϵ is a constant; $\partial_\mu \epsilon = 0$ (39). This also means it is not a full gauge-theory, unlike the standard model. Needless to say, SUSY is developed as a locally-invariant gauge theory, and also extended to a fully covariant form, i.e. by use of covariant derivatives in place of partial derivatives, paving the way to take SUSY into the realm of general relativity in a theory known as *super-gravity* (what-else?).

The fields are all massless. This is consistent for the spinor field χ since it is left-chiral and applicable only to massless particles. Furthermore, for scalar fields such as ϕ, there is no such consistency requirement, and a mass term such as $m^2\phi\phi^\dagger$ could easily be added to give a massive boson field ϕ. For addition of a spinor field, extra attention is required on the spinor representation. However, mass was introduced right at the start of SUSY [6], so the introduction of mass is not a major issue.

There are no interaction terms between the fields. The remedy is, of course, to add such terms, and it appears, once again, that Wess and Zumino [6] addressed this matter.

Consideration has only been given to a Lagrangian with a single spinor field χ (including its conjugate), hence too a single 'N=1' charge Q (plus conjugate). Although not discussed in detail, this means there are only two states, i.e. a single fermion and boson with equal mass, differing only in their spin - see the Introduction. This single-charge theory is known as *N=1 SUSY*. Needless to say, the introduction of additional charges, $Q_N, N > 1$ (plus their conjugate) gives higher-weight multiplet states. The $N > 1$ cases are collectively known as *extended super-symmetry.*

There are no auxiliary fields and therefore the simplified Lagrangian is limited to on-shell solutions only. The inclusion of an auxiliary field was discussed in Section (1.6), and it was noted that there is a more physically meaningful replacement of the field with a *super-potential*, see [2].

The above gives a few major additions required to go beyond the simple Lagrangian presented herein. The full Wess-Zumino Lagrangian is given considerable treatment in [2], as is the wider field of SUSY extended to the concepts of a *superspace*, whereby the spacetime coordinates are augmented with additional Grassman coordinates. Indeed, the entire book lays extensive foundation for advanced study, and the reader is therefore referred to the references given in both [2] and [3] to go beyond this.

2 Summary and Conclusions

This article started with a short overview and justification of why, despite no experimental evidence almost 50 years on, there is still merit in pursuing the mathematical development of SUSY. A key prediction of SUSY is that every fermion has a boson super-partner, and vice versa, with the same mass. A second key point is that the algebra of the charges (generators) is that of anti-commutation, and the components of its basic spinor representation are anti-commuting Grassman numbers. The basics of a full super-symmetric theory are then illustrated by way of a simplified form of the Wess-Zumino Lagrangian, comprising both bosonic and fermionic scalar fields. Standard variational techniques, using a global (not local) variation that mixes the two fields, are then used to show the action is stationary, in line with standard QFT. This then justifies the further development of the theory, with the first steps being the determination of the commutation algebra of the super-charges, particularly with regard to the charges of the Poincare spacetime algebra. This latter aspect is stated rather than justified, with the reader referred to the literature for full justification. Likewise, for the determination of the Lagrangian's Noether currents, which can be used to determine explicit representations of the super-charges.

In conclusion, although this article is a relatively basic illustration of super-symmetry, it shows that the basics of SUSY fit within both the existing mathematics of the standard model, plus it shows SUSY offers new ideas to overcome some current limitations. it is hard to deny that, despite lack of experimental evidence, SUSY offers a way forward in unification of gravity with the standard model, not least since it underpins super-string theory and super-gravity. Furthermore, history shows that nature has a predisposition to adhere to pure mathematical theory no matter how abstract it might at first seem, e.g. the group-theoretic approach to the standard model. In short, until a mathematical inconsistency is found, SUSY is worth pursuing.

3 References

References

[1] G. Junker, *Supersymmetric Methods in Quantum and Statistical Physics*, Springer, ISBN 3-540-61591-1

[2] Labelle, Patrick, *Supersymmetry Demystified*, McGraw Hill 2010, ISBN 978-0-07-163641-4

[3] Lewis H Ryder, *Quantum Field Theory*, Cambridge Univ. Press, Second Edition, ISBN 978-0-521-47814-6

[4] Roger Penrose, *The Road to Reality*, Jonathon Cape 2004, ISBN 978-0-224-04447-8

[5] J. Wess, B. Zumino, Supergauge Transformations in Four Dimensions, Nucl. Phys B70 (1974) 39-50

[6] J. Wess, B. Zumino, Supergauge Invariant Extension of Quantum Electrodynamics, Nucl. Phys B78 (1974) 1-13

[7] L O'Raifertaigh, *Group Structure of Gauge Theories*, Cambridge Monographs on Mathematical Physics, ISBN 0-521-34785-8

4 Appendices

4.1 Mass Dimension

The *mass dimension*, applicable when using natural units, is given below as a reference source for commonly used quantities in this article.

$$\begin{aligned}
&[M] = 1 \qquad (58)\\
&[E] = 1, \quad (E = Mc^2, c = 1)\\
&[T] = -1, \quad (E = h\nu, h = 1,\ \nu = T^{-1})\\
&[L] = -1, \quad (L = ct, c = 1)\\
&[P] = 1, \quad (P = E/C = m, c = 1)\\
&[\partial_\mu] = [\partial^\mu] = 1, \quad (\partial_\mu = 1/L)\\
&[\phi] = [\phi^\dagger] = 1\\
&[\partial_\mu \phi] = [\partial^\mu \phi] = 2\\
&[\sigma], [\bar{\sigma}] = 0\\
&[\epsilon] = [\epsilon^\dagger] = -\frac{1}{2}\\
&[\chi] = [\chi^\dagger] = \frac{3}{2}\\
&[\partial_\mu \chi] = \frac{5}{2}
\end{aligned}$$

4.2 Spinor Inner Products

The use of inner products in the main text is sparing, but the concept is very important since, of course, inner products should produce scalar invariants and, indeed, there are two, unique, *Lorentz-invariant* inner products between left chiral spinors η and χ defined as the follows:

$$\eta \cdot \chi = \eta^T(-i\sigma^2)\chi \tag{59}$$

$$\bar{\eta} \cdot \bar{\chi} = \eta^\dagger(i\sigma^2)\chi^{\dagger T} \tag{60}$$

The reader is referred to Ryder [3] for a thorough exposition of how left and right (AKA *type 1* and *Type II*) spinors transform under relativistic boosts and rotations, i.e. under transformations of the Lorentz group).

Aside. On a philosophical note, the fact that a classical, relativistic four-vector can have a non-classical, quantum spinor representation (two-fold) seems rather remarkable to the author and, whilst this is due to the Lorentz group having an $SU2(2) \otimes SU(2)$ representation, it seems almost too good to be true! There are some finer points though, see Ryder [3]. The point of this aside is that, once again, the mathematics is giving some very profound physical implications, and another reason why SUSY may not be a flight of mathematical fancy.

The quantity $i\sigma^2$ (where σ^2 is the second Pauli matrix, and not σ-squared), sandwiched between the two spinors in the above inner product definition, acts as a metric in spinor algebra, effectively raising the index so that the inner product has a similiar form to that of tensor contraction, e.g. as in $u \cdot v = u^\mu v_\mu = g^{\nu\mu} u_\nu v_\mu$. In SUSY this metric is often given the symbol ϵ, as in $\epsilon = \pm i\sigma_2$, where the sign choice depends upon convention. However, in the main text, ϵ denotes a ‘small’ (infinitesimal) spinor.

The over-struck bar on the *barred* spinors $\bar{\eta}, \bar{\chi}$ in the second inner product effectively means complex-conjugate, whilst the transposed transpose-conjugate of a spinor, e.g. $\chi^{\dagger T}$, is both a transpose of the spinor from a column to row spinor (or vice-versa), and a transpose conjugate of its elements, i.e.

$$\chi^\dagger = \begin{pmatrix}\chi_1^\dagger \chi_2^\dagger\end{pmatrix}, \quad \chi^{\dagger T} = \begin{pmatrix}\chi_1^\dagger \\ \chi_2^\dagger\end{pmatrix}, \tag{61}$$

Whilst it may seem easier to just write $\chi^{\dagger T}$ as simply χ^*, the spinor components may themselves be field operators, in which case a complex conjugation is not the same as the transpose conjugate. However, because this article only considers complex spinor components, then the transpose conjugate is the same as the complex conjugate.

Using the following standard definitions

$$\sigma_2 = \begin{pmatrix}0 & -i \\ i & 0\end{pmatrix}, \chi = \begin{pmatrix}\chi_1 \\ \chi_2\end{pmatrix}, \eta = \begin{pmatrix}\eta_1 \\ \eta_2\end{pmatrix} \tag{62}$$

then in component form the first inner product evaluates to

$$\eta \cdot \chi = \eta_2\, \chi_1 - \eta_1 \chi_2 \tag{63}$$

Grassman numbers [8] . Whilst the above makes spinor components appear to be complex, they are in fact Grassman numbers that anti-commute, i.e. two Grassman numbers a and b are such that $ab = -ba$, or in anti-commutation form $\{a, b\} = 0$, noting that Grassman numbers do commute with complex numbers, and hence too reals. This anti-commutation property also means that $a^2 = b^2 = 0$.

This Grassman property of spinor components means that they are more than just plain complex numbers, and so if $\eta = \chi$ in the above inner product then

$$\chi \cdot \chi = \chi_2\, \chi_1 - \chi_1 \chi_2 = 2\, \chi_2\, \chi_1 = -2\, \chi_1\, \chi_2 \tag{64}$$

The second inner product, $\bar{\eta} \cdot \bar{\chi}$ above, relating barred spinors, looks like the complex conjugate of the first since $\chi^{\dagger T} = \chi^*$ (above) and $\eta^{T\dagger} = \eta^*$, but the quantity $i\sigma^2$ is actually a real-valued matrix so its conjugate $(i\sigma^2)^* = i\sigma^2$ In fact, the two inner products relate simply as follows:

$$(\eta \cdot \chi)^* = -\bar{\eta} \cdot \bar{\chi} \tag{65}$$
$$(\eta \cdot \chi)^\dagger = \bar{\eta} \cdot \bar{\chi} \tag{66}$$

where the transpose conjugate in the second relation has the effect of swapping the spinor multiplication order, and introduces a minus due to spinor anti-commutation, that is cancelled by the minus from the conjugate part of the transpose conjugate. Hence the transpose conjugate of the inner product leaves it invariant, but the simpler complex conjugate in the first inner product (with no transposition and hence no spinor multiplication swap), introduces a minus with no compensating minus since the spinor multiplication ordering is unchanged.

Lastly, the barred notation is not really required herein but may well be seen in SUSY literature when used in a wider context. Consequently, its use in this article is kept to almost zero, and the barred form of inner product, as above, will always be expanded in full using its unbarred spinor definition.

[8] Hermann Grassmann 1809-1877, See Penrose [4]